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MECHANICS BY QUATERNIONS.

BY PROF. E. W. HYDE, UNIVERSITY OF CINCINNATI.

(12) RETURNING now to equations (22) and (23), let us first find the condition that the system of forces may have a single resultant. Evidently if ${}_{,}G$ be perpendicular to ${}_{,}R$ the forces composing ${}_{,}G$ may be taken as parallel to ${}_{,}R$, and we shall then have a system of three parallel forces whose sum is not zero, which may therefore be combined into a single resultant.

Hence the required condition is

$$S{}_{,}R{}_{,}G = S{}_{,}R{}_{,}G_0 = 0. \tag{24}$$

This equation is satisfied by ${}_{,}R = 0$ and ${}_{,}G = 0$, as well as by ${}_{,}R$ perpendicular to ${}_{,}G$. If the first is true there is no resultant, and the system is equivalent to a couple only.

If ${}_{,}G = {}_{,}G_0 - V{}_{,}\pi{}_{,}R = 0$, the locus of ${}_{,}\pi$ will be the locus of all points at which there is a single resultant. ${}_{,}G_0$ being a constant vector this is the equation of a straight line as appears by solving the equation for ${}_{,}\pi$. Since the Scalar part of the quaternion ${}_{,}\pi{}_{,}R$ is indeterminate call it x, then

$${}_{,}\pi{}_{,}R = x + V{}_{,}\pi{}_{,}R = x + {}_{,}G_0\,;$$

$$\therefore \qquad {}_{,}\pi = x{}_{,}R^{-1} + {}_{,}G_0{}_{,}R^{-1}. \tag{25}$$

The last term of this equation is a vector because

$$S{}_{,}G_0{}_{,}R^{-1} = S{}_{,}R^{-1}\,V{}_{,}\pi{}_{,}R = 0.$$

It is also evident that if ${}_{,}G$ be any constant vector whatever the locus of ${}_{,}\pi$ is a right line parallel to ${}_{,}R$, i. e., the moment is the same about all p'ts of any line parallel to ${}_{,}R$.

(13). We will next find the locus of ${}_{,}\pi$ when $G = T{}_{,}G$ is a constant quantity; i. e., $T({}_{,}G_0 - V{}_{,}\pi{}_{,}R) = G =$ constant, or squaring and transposing, and remembering that $V^2{}_{,}\pi{}_{,}R = S^2{}_{,}\pi{}_{,}R - {}_{,}\pi^2{}_{,}R^2$,

$${}_{,}\pi^2{}_{,}R^2 - S^2{}_{,}\pi{}_{,}R + 2S{}_{,}G_0{}_{,}\pi{}_{,}R = G^2 - G_0^2. \tag{26}$$

This is easily seen to be the equation of a circular cylinder whose axis is parallel to ${}_{,}R$.

To find the position of the axis, let us change the origin by putting ${}_{,}\pi + {}_{,}\delta$ for ${}_{,}\pi$, with the condition $S{}_{,}\pi{}_{,}R = 0$; so that ${}_{,}\delta$ is the vector of the point where the axis pierces the plane $S{}_{,}\pi{}_{,}R = 0$. Substituting in (26) with this condition we have

$$({}_{,}\pi + {}_{,}\delta)^2{}_{,}R^2 + 2S{}_{,}G_0({}_{,}\pi + {}_{,}\delta){}_{,}R = G^2 - G_0^2.$$

If the new origin is to be at a point of the axis, the terms of the 1st degree in ${}_{,}\pi$ must disappear; therefore

$${}_{,}R^2S{}_{,}\pi{}_{,}\delta + S{}_{,}G_0{}_{,}\pi{}_{,}R = 0 = S({}_{,}R^2{}_{,}\delta - V{}_{,}G_0{}_{,}R){}_{,}\pi\,;$$

$$\therefore \qquad {}_{,}\delta = {}_{,}R^{-2}\,V{}_{,}G_0{}_{,}R = V{}_{,}G_0{}_{,}R^{-1}. \tag{27}$$

The equation of the axis of the cylinder is then

$${}_{,}\rho = V{}_{,}G_0{}_{,}R^{-1} + x{}_{,}R. \qquad (28)$$

Let us next fiind the value, or rather the locus, of ${}_{,}\pi$ when $G = T{}_{,}G$ is a minimum. The value of G increases with π to ∞, so that there is no maximum. If G is a minimum we must have

$$S{}_{,}Gd{}_{,}G = 0, \text{ i. e., } S({}_{,}G_0 - V{}_{,}\pi{}_{,}R)V{}_{,}Rd{}_{,}\pi = 0.$$

Expanding and reducing we have

$$S(V{}_{,}G_0{}_{,}R - {}_{,}\pi{}_{,}R^2 + {}_{,}RS{}_{,}\pi{}_{,}R)d{}_{,}\pi = 0,$$

which can only be justisfied in general by equating the quantity in parentheses to zero. Therefore

$${}_{,}\pi{}_{,}R^2 = V{}_{,}G_0{}_{,}R + {}_{,}RS{}_{,}\pi{}_{,}R$$

or

$${}_{,}\pi = V{}_{,}G_0{}_{,}R^{-1} + R^{-1}S{}_{,}\pi{}_{,}R. \qquad (29)$$

This is the equation of a straight line, and is identical with equation (28) and also with eq. (25), which is the particular case when $S{}_{,}G_0{}_{,}R^{-1} = 0$.

The identity between (28) and (29) will be evident on replacing ${}_{,}\rho$ in the former by ${}_{,}\pi$, and then operating by $S.{}_{,}R$, when we shall find

$$x = {}_{,}R^{-2}S{}_{,}\pi{}_{,}R.$$

It appears therefore that the axis of the cylinder $T{}_{,}G = G =$ const. is the locus of ${}_{,}\pi$ when G is a minimum; and that if the system of forces is capable of reduction to a single resultant, the line of action of this resultant is this same axis, i. e., in this case, the minimum value of G is zero.

This axis is called the *Central Axis* of the system. If we substitute the value of ${}_{,}\pi$ from (29) in the value of ${}_{,}G$ we have

$${}_{,}\mathrm{G} = {}_{,}\mathrm{G}_0 - V{}_{,}\pi{}_{,}R = {}_{,}\mathrm{G}_0 - V(V{}_{,}\mathrm{G}_0{}_{,}R^{-1} + {}_{,}R^{-1}S{}_{,}\pi{}_{,}R){}_{,}R$$
$$= {}_{,}\mathrm{G}_0 + V.{}_{,}R\,V{}_{,}\mathrm{G}_0{}_{,}R^{-1} = {}_{,}\mathrm{G}_0 + {}_{,}R^{-1}S{}_{,}R{}_{,}\mathrm{G}_0 - {}_{,}\mathrm{G}_0 = {}_{,}R^{-1}S{}_{,}R{}_{,}\mathrm{G}_0, \qquad (30)$$

for the value of ${}_{,}\mathrm{G}$ when G is a minimum. Eq. (30) shows that in this case ${}_{,}\mathrm{G}$ is a vector parallel to ${}_{,}R$, i. e., the plane of the resultant couple is perpendicular to ${}_{,}R$. Let us see if there is any other position of ${}_{,}R$ for which ${}_{,}\mathrm{G}$ is parallel to ${}_{,}R$, i. e., $V{}_{,}\mathrm{G}{}_{,}R = 0$. Operating on ${}_{,}\mathrm{G}$ by $V{}_{,}R$, we have $V{}_{,}R{}_{,}\mathrm{G}_0 - V.{}_{,}R\,V{}_{,}\pi{}_{,}R = 0$, $\therefore$ $V{}_{,}R{}_{,}\mathrm{G}_0 - {}_{,}RS{}_{,}\pi{}_{,}R + {}_{,}\pi{}_{,}R^2 = 0$, or

$${}_{,}\pi = V{}_{,}\mathrm{G}_0{}_{,}R^{-1} + {}_{,}R^{-1}S{}_{,}\pi{}_{,}R,$$

which is (29) again, so that the central axis is the only line such that when the resultant acts along it the plane of the resultant couple is perp. to ${}_{,}R$.

(14). Let us determine the conditions that must hold in order that the resolved parts along any straight line whatever of a system of forces acting on a rigid body may be in equilibrium.

Let ε be a unit vector in any direction, then $\varepsilon^{-1}S\varepsilon{}_{,}P$ is the resolved part of ${}_{,}P$ along ε. In equation (22) and (23) put $\varepsilon^{-1}S\varepsilon{}_{,}P$ for ${}_{,}P$; therefore

$$\Sigma[\varepsilon^{-1}S\varepsilon{}_{,}P] = \varepsilon^{-1}S\varepsilon\Sigma({}_{,}P) = 0, \therefore S\varepsilon\Sigma({}_{,}P) = 0,$$

and as this must hold for all directions of ε

$$\Sigma({}_{,}P) = {}_{,}R = 0. \qquad (31)$$

Also $$\Sigma[V{}_{,}\rho\varepsilon^{-1}S\varepsilon{}_{,}P] = 0 = V.\varepsilon\Sigma[{}_{,}\rho S\varepsilon{}_{,}P]. \qquad (32)$$

If this is to be true for *all* directions of ε we must have

$$n\varepsilon = \Sigma[{}_{,}\rho S\varepsilon{}_{,}P]. \qquad (33)$$

Now the general value of ε in terms of any three non-coplanar vectors ε_1, ε_2, ε_3 is

$$\varepsilon = \frac{1}{S\varepsilon_1\varepsilon_2\varepsilon_3}\Big[\varepsilon_1 S\varepsilon_2\varepsilon_3\varepsilon + \varepsilon_2 S\varepsilon_3\varepsilon_1\varepsilon + \varepsilon_3 S\varepsilon_1\varepsilon_2\varepsilon\Big],$$

and hence $\Sigma[{}_{,}\rho S\varepsilon{}_{,}P]$ must be capable of reduction to such a form if (33) is to be satisfied.

(15). If a rigid body be in equilibrium under the action of a system of forces and if equilibrium be maintained when the body has been turned through the angle θ about any axis (θ being not a multiple of two right angles, and the system of forces being constant in magnitude and direction), then the system will be in equilibrium whatever value θ may have.

We have the conditions

$$\Sigma({}_{,}P) = 0 \text{ and } \Sigma(V{}_{,}\rho{}_{,}P) = 0. \qquad (34)$$

Let ε be a unit vector along the axis about which the body is to be revolved, then every ${}_{,}\rho$ will be changed to $\varepsilon^{\theta\div\pi}{}_{,}\rho\varepsilon^{-\theta\div\pi}$. (See Tait's Quaternions, Art. 351.) Substituting in the second of (34) we have

$$\Sigma[V\varepsilon^{\theta\div\pi}{}_{,}\rho\varepsilon^{-\theta\div\pi}{}_{,}P] = 0,$$

$$\text{i. e. } \Sigma[V(\cos\tfrac{1}{2}\theta + \varepsilon\sin\tfrac{1}{2}\theta){}_{,}\rho(\cos\tfrac{1}{2}\theta - \varepsilon\sin\tfrac{1}{2}\theta){}_{,}P] = 0.$$

Expanding, reducing and making use of (34) this becomes

$$2\cos\tfrac{1}{2}\theta[\Sigma(\varepsilon S{}_{,}\rho{}_{,}P) - \Sigma({}_{,}\rho S\varepsilon{}_{,}P)] - V\varepsilon\Sigma({}_{,}\rho S\varepsilon{}_{,}P - {}_{,}PS\varepsilon{}_{,}\rho)\sin\tfrac{1}{2}\theta = 0. \quad (35)$$

This being an equation between two vectors multiplied by scalar coefficients can only be satisfied by making both coefficients zero, or else both vectors zero. As the coefficients cannot be zero together the vectors must be;

$$\therefore \Sigma(\varepsilon S{}_{,}\rho{}_{,}P) = \Sigma({}_{,}\rho S\varepsilon{}_{,}P), \qquad (36)$$

$$V\varepsilon\Sigma({}_{,}\rho S\varepsilon{}_{,}P) = V\varepsilon\Sigma({}_{,}PS\varepsilon{}_{,}\rho). \qquad (37)$$

Eliminate $\Sigma({}_{,}\rho S\varepsilon{}_{,}P)$ between (36) and (37),

$$V\varepsilon\Sigma(\varepsilon S{}_{,}\rho{}_{,}P) = V\varepsilon\Sigma({}_{,}PS\varepsilon{}_{,}\rho);$$

the first member is zero because $V\varepsilon\varepsilon = 0$, $\therefore$ the second member is also zero. We have supposed (35) to be true for *some* value of θ, hence the conditions (36) and (37) must be satisfied, and if they *are* satisfied (35) is true for *any* value of θ, Q. E. D.

(16). We will now show how the general equations of equilibrium of a rigid body may be deduced from those for forces acting on a particle.

Suppose a system of n rigidly connected particles. The action between any two particles to be taken along the the straight line joining them, and to be mutual, i. e., the action of the first on the second equal and opposite to that of the second on the first. Let ${}_{,}P_1$ be the external force acting on the particle 1 and let Q_1, Q_2 &c., be the forces acting between it and other particles. Then there will be n equations

$$ {}_{,}P_1+\Sigma({}_{,}Q') = 0,\ {}_{,}P_2+\Sigma({}_{,}Q'') = 0, \text{ etc.,} $$

where the primes indicate a different set of ${}_{,}Q$'s in each case. From the hypotheses above however it is clear that for every ${}_{,}Q$, as ${}_{,}Q_i$, we shall have also a force $-{}_{,}Q_i$; hence if we add up the equations above, the ${}_{,}Q$'s will all cancel out and leave us

$$ \Sigma({}_{,}P) = 0. \tag{38} $$

Let ${}_{,}\rho_1$, ${}_{,}\rho_2$ &c., be the vectors of the particles; then from the equations above we have

$$ V_{,}\rho_{1,}P_1+V_{,}\rho_1\Sigma({}_{,}Q') = 0,\ V_{,}\rho_{2,}P_2+V_{,}\rho_2\Sigma({}_{,}Q'') = 0, \text{ etc.} $$

On adding these equations it appears from what has been already shown that we shall have a series of terms containing Q's such as

$$ V_{,}\rho_{l,}Q_{lm}-V_{,}\rho_{m,}Q_{lm} = V({}_{,}\rho_l-{}_{,}\rho_m)_{,}Q_{lm}, $$

in which ${}_{,}\rho_l$ and ${}_{,}\rho_m$ are the vectors of the lth and mth particles and ${}_{,}Q_{lm}$ is the force between them. But as, by hypothesis, ${}_{,}Q_{lm}$ acts along ${}_{,}\rho_l-{}_{,}\rho_m$, it follows that every such vector is zero. Hence on summing we shall have

$$ \Sigma(V_{,}\rho_{,}P) = 0. \tag{39} $$

(17). Some additional illustrative examples will now be given.

Ex. 1. A cube is acted upon by 4 forces; one is in a diagonal, and the others in edges no two of which are in the same plane, and which do not meet the diagonal: find the conditon that there may be a single resultant.

Let i, j, k be the unit vector edges of the cube, and let the four forces be P_1i, P_2j, P_3k and $P_4\,U(i+j+k)$;

$$ \therefore {}_{,}R = P_1i+P_2j+P_3k+P_4\,U(i+j+k). $$

Take moments about the origin so that ${}_{,}\pi=0$, the origin being at a corner of the cube;

$$ \begin{aligned} \therefore {}_{,}G &= VjP_1i+VkP_2j+ViP_3k \\ &= -iP_2-jP_3-kP_1. \end{aligned} $$

The condition is by eq. (24), $S_{,}R_{,}G = 0$; therefore

$$ S[iP_2+jP_3+kP_1][iP_1+jP_2+kP_3+P_4\,U(i+j+k)] = 0. $$

But
$$ U(i+j+k) = \frac{i+j+k}{T(i+j+k)} = \frac{1}{\sqrt{3}}(i+j+k); $$

$$ \therefore S[iP_2+jP_3+kP_1][iP_1+jP_2+kP_3+P_4\sqrt{\tfrac{1}{3}}\cdot(i+j+k)] = 0 $$

$$\therefore P_1P_2+P_2P_4\surd\tfrac{1}{3}+P_2P_3+P_3P_4\surd\tfrac{1}{3}+P_3P_1+P_1P_4\surd\tfrac{1}{3}=0;$$
$$\therefore P_4(P_1+P_2+P_3)+\surd 3.(P_1P_2+P_2P_3+P_3P_1)=0.$$

Ex. 2. Six equal forces act along the edges of a cube which do not meet a given diagonal taken in order; find the resultant action.

Let P be the tensor of each force, i, j, k, unit vectors along the edges, the length of each edge being a

$$\therefore {}_{,}R=P(i+j+k-i-j-k)=0,$$

and taking moments about a corner at the extremity of the above mentioned diagonal,

$$\begin{aligned}{}_{,}\mathrm{G}={}_{,}\mathrm{G}_0&=Pa\,V(-ij+ji+ik-ki-jk+kj)=-2Pa(i+j+k)\\&=-2\surd 3.Pa\,U(i+j+k).\end{aligned}$$

Ex. 3. Find the central axis of two forces ${}_{,}P$ and ${}_{,}Q$.

Let the perpendicular distance between the forces be $2{}_{,}a$ and take the origin at its middle point, so that $S{}_{,}a{}_{,}P=S{}_{,}a{}_{,}Q=0$, and

$${}_{,}\mathrm{G}={}_{,}\mathrm{G}_0-V{}_{,}\pi{}_{,}R=V{}_{,}a({}_{,}P-{}_{,}Q)-V{}_{,}\pi({}_{,}P+{}_{,}Q).$$

Then, by eq (29), ${}_{,}\pi=-V.({}_{,}P+{}_{,}Q)^{-1}\,V{}_{,}a({}_{,}P-{}_{,}Q)+({}_{,}P+{}_{,}Q)^{-1}S{}_{,}\pi({}_{,}P+{}_{,}Q)$

$$=\frac{-1}{({}_{,}P+{}_{,}Q)^2}\Big[V.({}_{,}P+{}_{,}Q)\,V{}_{,}a({}_{,}P-{}_{,}Q)-({}_{,}P+{}_{,}Q)S{}_{,}\pi({}_{,}P+{}_{,}Q)\Big]$$

$$=\frac{{}_{,}P^2-{}_{,}Q^2}{({}_{,}P+{}_{,}Q)^2}.{}_{,}a+{}_{,}R^{-1}S{}_{,}\pi{}_{,}R=\frac{P^2-Q^2}{P^2+Q^2-2S{}_{,}P{}_{,}Q}.{}_{,}a+{}_{,}R^{-1}S{}_{,}\pi{}_{,}R.$$

If the forces are perpendicular $S{}_{,}P{}_{,}Q=0$, and we have

$${}_{,}\pi=\frac{P^2-Q^2}{P^2+Q^2}.{}_{,}a+{}_{,}R^{-1}S{}_{,}\pi{}_{,}R.$$

In this case the distances along ${}_{,}a$ from the central axis to P and Q are respectively ${}_{,}a.\frac{2Q^2}{P^2+Q^2}$ and ${}_{,}a.\frac{2P^2}{P^2+Q^2}$, so that these distances bear to each other the ratio of Q^2 to P^2.

Ex. 4. Forces act at the angular points of a tetrahedron in directions respectively perpendicular to the opposite faces, and proportional to the areas of the faces in magnitude: show that the forces have the property considered in Art. 14.

Let ${}_{,}a$, ${}_{,}b$, ${}_{,}c$ be three vector edges of the tetrahedron measured out from one vertex taken as the origin; then the forces may be represented by $V{}_{,}a{}_{,}b$, $V{}_{,}b{}_{,}c$, $V{}_{,}c{}_{,}a$ and $V({}_{,}c-{}_{,}a)({}_{,}b-{}_{,}a)$. By eq. (32) we must have

$$V.\varepsilon\Sigma({}_{,}\rho S\varepsilon{}_{,}P)=0,\ \Sigma({}_{,}\rho S\varepsilon{}_{,}P)={}_{,}aS\varepsilon{}_{,}b{}_{,}c+{}_{,}bS\varepsilon{}_{,}c{}_{,}a+{}_{,}cS\varepsilon{}_{,}a{}_{,}b=\varepsilon S{}_{,}a{}_{,}b{}_{,}c;$$

therefore the condition is satisfied.

(18). *Constrained Body.*—First let a rigid body have one fixed point, and let this point exert a force ${}_{,}F$ against the body; then for equilibrium,

$$\Sigma({}_{,}P)+{}_{,}F={}_{,}R+{}_{,}F=0, \tag{40}$$

and, if we take the origin at the fixed point,

$$ {}_{,}\mathrm{G}_0=\Sigma(V{}_{,}\rho{}_{,}P)=0. \tag{41}$$

Eq. (40) gives the pressure on the fixed point and (41) is the condition of equilibrium.

The system therefore reduces to a single resultant acting through this pt.

Second, let there be two fixed points in the body, and let the forces they exert on the body be respectively ${}_{,}F_1$ and ${}_{,}F_2$. Since two points determine a right line, it is evident that all points in the right line joining the two fixed points will be incapable of motion, and the only condition of equilibrium is that the resultant moment about this line shall be zero. Let ε be a unit vector along the fixed line, and let the origin be taken on this line at distances a and b from the fixed points. Then the general equations (22) and (23) give

$$\Sigma({}_{,}P)+{}_{,}F_1+{}_{,}F_2=0, \tag{42}$$

$$V\Sigma({}_{,}\rho{}_{,}P)+aV\varepsilon{}_{,}F_1+bV\varepsilon{}_{,}F_2=0. \tag{43}$$

Operate on (43) by $S.\varepsilon$, $\therefore$

$$S\varepsilon\Sigma({}_{,}\rho{}_{,}P)=0=S\varepsilon{}_{,}G_0. \tag{44}$$

Eq. (44) is the condition of equilibrium, for it makes the axis of the resultant couple perpendicular to ε, so that there can be no moment about ε. By elimination between (42) and (43) we find

$$\left.\begin{array}{l} V\varepsilon{}_{,}F_1=\dfrac{1}{a-b}\Big[bV\varepsilon\Sigma({}_{,}P)-{}_{,}\mathrm{G}_0\Big],\quad V\varepsilon{}_{,}F_2=\dfrac{-1}{a-b}\Big[aV\varepsilon\Sigma({}_{,}P)-{}_{,}\mathrm{G}_0\Big]. \\ \text{But } {}_{,}F_1=\varepsilon^{-1}S\varepsilon{}_{,}F_1+\varepsilon^{-1}V\varepsilon{}_{,}F_1 \text{ and } {}_{,}F_2=\varepsilon^{-1}S\varepsilon{}_{,}F_2+\varepsilon^{-1}V\varepsilon{}_{,}F_2, \end{array}\right\} \tag{44a}$$

hence the components of ${}_{,}F_1$ and ${}_{,}F_2$ along ε are indeterminate, as $S\varepsilon{}_{,}F_1$ and $S\varepsilon{}_{,}F_2$ are subject only to the condition derived from (42),

$$S\varepsilon{}_{,}F_1+S\varepsilon{}_{,}F_2+S\varepsilon\Sigma({}_{,}P)=0. \tag{44b}$$

If the body is capable of sliding along the line ε as well as of turning around it, we must have the additional condition of equilibrium,

$$S\varepsilon\Sigma({}_{,}P)=0, \tag{45}$$

which is equivalent to equation (3).

If *three* points of the body, not in one straight line, are fixed, it can have no motion either of rotation or translation, and the three vectors of the fixed points are indeterminate.

(19). *Body resting against a smooth plane.*—First let the body rest on one point, and call the force exerted by the plane against the body $N\varepsilon$, ε being a unit vector perpendicular to the plane. Then if we take the origin at the point where the body rests on the plane, and take moments about this point for equilibrium,

$$\Sigma({}_{,}P)+N\varepsilon=0, \tag{46}$$

$${}_{,}\mathrm{G}_0=\Sigma(V{}_{,}\rho{}_{,}P)=0. \tag{47}$$

As this gives $S{}_{,}R{}_{,}\mathrm{G}=S\Sigma({}_{,}P){}_{,}\mathrm{G}_0=0$, which by eq. (24) is the condition for a single resultant, it follows that the system reduces to a single force ${}_{,}R=-N\varepsilon$ perpendicular to the plane, and acting through the point of contact. The latter follows from equation (25).

Second, let the body rest on *two* points, and let $N_1\varepsilon$ and $N_2\varepsilon$ be the normal reactions of the plane against the body. Then

$${}_{,}R+N_1\varepsilon+N_2\varepsilon=\Sigma({}_{,}P)+\varepsilon(N_1+N_2)=0, \tag{48}$$

$$\Sigma(V{}_{,}\rho{}_{,}P)+N_1V{}_{,}a_1\varepsilon+N_2V{}_{,}a_2\varepsilon={}_{,}G_0+V(N_1{}_{,}a_1+N_2{}_{,}a_2)\varepsilon=0, \tag{49}$$

provided that moments are taken about the origin, and ${}_{,}a_1$ and ${}_{,}a_2$ are the vectors of the points of contact. We have at once from (48) $U{}_{,}R=-\varepsilon$, and $T{}_{,}R=R=\Sigma(P)=N_1+N_2$. Operating on (49) by $S.{}_{,}R=-RS\varepsilon$, we have

$$S\varepsilon{}_{,}\mathrm{G}_0=S\varepsilon\Sigma(V{}_{,}\rho{}_{,}P)=0, \tag{50}$$

which by (24) is the condition for a single resultant. Also operating on (49) by $S.{}_{,}a_1$ and $S.{}_{,}a_2$ we find

$$N_1=\frac{S{}_{,}a_2{}_{,}\mathrm{G}_0}{S{}_{,}a_1{}_{,}a_2\varepsilon},\qquad N_2=-\frac{S{}_{,}a_1{}_{,}\mathrm{G}_0}{S{}_{,}a_1{}_{,}a_2\varepsilon}. \tag{51}$$

The equation of the single resultant is by (25)

$${}_{,}\pi=\mathrm{R}^{-1}x\varepsilon-\mathrm{R}^{-1}\varepsilon{}_{,}\mathrm{G}_0,$$

that is ${}_{,}\mathrm{R}$ is applied at a distance from the origin, measured perpendicular to itself, of $-\mathrm{R}^{-1}\varepsilon{}_{,}\mathrm{G}_0$. From the equation just given ${}_{,}\mathrm{G}_0=\mathrm{R}V\varepsilon{}_{,}\pi$; substitute this value in (49) an we have

$$V(-{}_{,}\pi\mathrm{R}+N_1{}_{,}a_1+N_2{}_{,}a_2)\varepsilon=0;$$

$$\therefore {}_{,}\pi=\frac{N_1{}_{,}a_1+N_2{}_{,}a_2}{\mathrm{R}}=\frac{N_1{}_{,}a_1+N_2{}_{,}a_2}{N_1+N_2}, \tag{52}$$

so that the resultant ${}_{,}\mathrm{R}$ is in the same line with the resultant of $N_1\varepsilon$ and $N_2\varepsilon$, as must evidently be the case.

Third, suppose the body to rest on *three* points. Then as before

$$\Sigma({}_{,}P)+(N_1+N_2)e^{*}={}_{,}\mathrm{R}+(N_1+N_2+N_3)e=0, \tag{53}$$

$${}_{,}\mathrm{G}_0+V(N_1{}_{,}a_1+N_2{}_{,}a_2+N_3{}_{,}a_3)e=0; \tag{54}$$

$\therefore U{}_{,}\mathrm{R}=-e$, $T{}_{,}\mathrm{R}=\mathrm{R}=N_1+N_2+N_3$, $Se{}_{,}\mathrm{G}_0=Se\Sigma(V{}_{,}\rho{}_{,}P)=0$, as before, so that the system reduces to a single resultant. The equation of this resultant will be precisely the same as in the last case. We shall also have a value of ${}_{,}\pi$ like (52) with a term $N_3{}_{,}a_3$ added in the numerator and a term N_3 in the denominator. The value of N_1, found by elimination between (53) and (54), is

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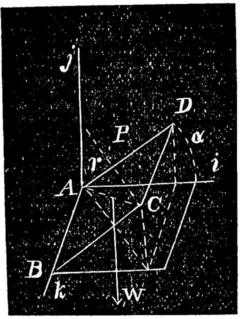
*Used here to represent ε.

$$N_1 = \frac{S({}_{,}a_2 - a_3){}_{,}\mathrm{G}_0 + \mathrm{R}S{}_{,}a_2\,{}_{,}a_2 e}{S({}_{,}a_1\,{}_{,}a_2 + {}_{,}a_2\,{}_{,}a_3 + {}_{,}a_3\,{}_{,}a_1)e}. \tag{55}$$

The values of N_2 and N_3 may be found from that of N_1 by cyclic permutation of subscripts.

If there are more than three points of contact the values of N_1, N_2 etc., become indeterminate, but we shall always have the condition for a single resultant, $Se{}_{,}G_0 = 0$.

(20). *Example.*— The lid *ABCD* of a cubical box, moveable about hinges at *A* and *B*, is held at a given angle a with the horizon by a horizontal string connecting *C* with a point vertically over *A*: find the pressure on each hinge.



We have, if the edge of the cube $= 1$,

$$\Sigma({}_{,}P) = -Wj - PU(i\cos a + k)$$

$$= -Wj - \frac{P}{\sqrt{(1+\cos^2 a)}}\left(i\cos a + k\right),$$

and $\mathrm{G}_0 = V[-\tfrac{1}{2}W(i\cos a + k)j - Pj\sin a.\,U(i\cos a + k)]$

$$= -\tfrac{1}{2}W(i\cos a + k)j - \frac{P\sin a}{\sqrt{(1+\cos^2 a)}}.j(i\cos a + k)$$

$$= \left(\frac{P\sin a}{\sqrt{(1+\cos^2 a)}} - \frac{W}{2}\right)\left(k\cos a - i\right).$$

The equation of condition (44) must be satisfied, therefore, substituting k for e, as the vector along the fixed line,

$$\left(\frac{P\sin a}{\sqrt{(1+\cos^2 a)}} - \frac{W}{2}\right)\cos a = 0; \text{ whence}$$

$$P = \frac{W\sqrt{(1+\cos^2 a)}}{2\sin a}.$$

Making use of this value of P we have

$$\Sigma({}_{,}P) = -Wj - \frac{W}{2\sin a}\left(i\cos a + k\right), \text{ and } {}_{,}G_0 = 0.$$

Hence by (44a) and (44b) we find, if ${}_{,}F_1$ acts at A and ${}_{,}F_2$ at B,

$${}_{,}F_1 = k^{-1}Sk{}_{,}F_1 + kV\left[-Wk\left(j + \tfrac{1}{2}i\cot a + \frac{1}{2\sin a}k\right)\right]$$

$$= \tfrac{1}{2}Wi\cot a + Wj - kSk{}_{,}F_1,$$

$${}_{,}F_2 = k^{-1}Sk{}_{,}F_2 = -kSk{}_{,}F_2; \text{ with the relation}$$

$$Sk{}_{,}F_1 + Sk{}_{,}F_2 + \tfrac{1}{2}W\operatorname{cosec} a = 0.$$

It is to be noted that the a and b of (43) are equal respectively to 0 and 1.